

Hadronic contribution to the muon $g - 2$: a Dyson-Schwinger perspective

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Abstract

We summarize our results for hadronic contributions to the anomalous magnetic moment of the muon (a_μ), the one from hadronic vacuum-polarisation (HVP) and the light-by-light scattering contribution (LBL), obtained from the Dyson-Schwinger equations (DSE's) of QCD. In the case of HVP we find good agreement with model independent determinations from dispersion relations for a_μ^{HVP} as well as for the Adler function with deviations well below the ten percent level. From this we conclude that the DSE approach should be capable of describing a_μ^{LBL} with similar accuracy. We also present results for LBL using a resonance expansion of the quark anti-quark T-matrix. Our preliminary value is $a_\mu^{\text{LBL}} = (217 \pm 91) \times 10^{-11}$.

1 Introduction

In the search for new physics beyond the standard model the anomalous magnetic moment of the muon (a_μ) is one of the most interesting observables. Compared to the corresponding electron anomaly (a_e) it is more sensitive to contributions from high lying scales. These include the weak interactions, QCD and potential new physics [1]. Especially the contributions from soft QCD deserve highest attention because, due to their non-perturbative nature and the resulting technical complications, they dominate the theoretical standard model (SM) prediction.

The efforts of the E821 experiment at Brookhaven National Lab [2, 3] as well as theoretical efforts of more than a decade [4] culminated in a determination of a_μ down to a level where significant deviations have been found

$$\text{Experiment: } 116\,592\,089(63) \times 10^{-11} \quad , \quad (1)$$

$$\text{Theory: } 116\,591\,828(49) \times 10^{-11} \quad , \quad (2)$$

where the theoretical number is taken from Ref. [5]. Comparing theory and experiment the deviation amounts to $a_\mu^{\text{exp}} - a_\mu^{\text{theo}} = 261(80)$ which corresponds to a 3.2σ effect. In order to confirm this result the uncertainties have to be reduced further.

There are two hadronic contributions that dominate the SM uncertainty. There is the hadronic vacuum polarisation contribution (HVP) which gives rise to the leading hadronic contribution as well as the leading SM uncertainty contribution [5]

$$a_\mu^{\text{HVP,DR}} = 6\,949.1(42.7) \times 10^{-11}. \quad (3)$$

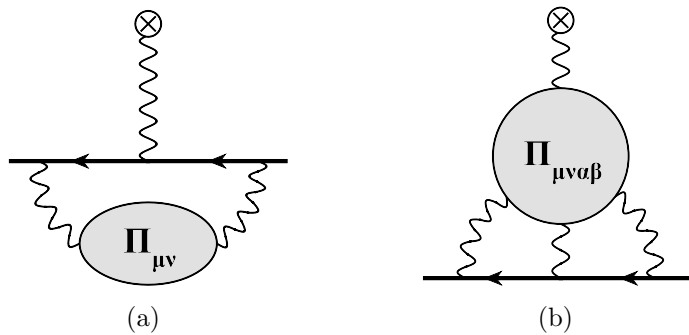


Figure 1: The two classifications of corrections to the photon-muon vertex function: (a) hadronic vacuum polarization contribution to a_μ . The vertex is dressed by the vacuum polarization tensor $\Pi_{\mu\nu}$; (b) the hadronic light-by-light scattering contribution to a_μ .

The relevant diagram, involving the hadronic one-particle irreducible (1PI) photon self-energy $\Pi_{\mu\nu}$ is shown in figure 1(a). The next-to leading uncertainty contribution comes from the light-by-light (LBL) scattering contribution that is shown in fig. 1(b). Estimates from the viewpoint of effective field theory (EFT) from different approaches were recently combined into a single number [6]

$$a_\mu^{\text{LBL}} = 105(26) \times 10^{-11}. \quad (4)$$

The uncertainty given here is rather small compared to most estimates. In fact our results indicate that this error may be far too optimistic.

Our strategy to determine these quantities is the following. We work with the Dyson-Schwinger and Bethe-Salpeter equations (DSE/BSE) of QCD [7, 8]. With these we calculate the HVP contribution to a_μ where we use a parameter set, among others, that is completely fixed by meson phenomenology. The HVP contribution can be compared to essentially model independent result from dispersion relations [9] such that the calculation serves as a non-trivial cross check of our methods. Afterwards we approach the LBL contribution using exactly the same truncation such that we have reasons to believe that we can ultimately reach a similar precision as in the case of HVP.

This proceedings contribution is organized as follows. First of all we summarise the employed truncation in section 2. The HVP contribution will be discussed in section 3 and LBL in sec. 4. Afterwards we discuss our results for both of these contributions in section 5 and conclude.

2 Calculational scheme

We work in rainbow-ladder truncation of QCD using the Maris-Tandy model of the quark-gluon interaction [10]. The central object in this approach is the quark DSE

$$S^{-1}(p) = Z_2 S_0^{-1} + Z_2^2 \frac{4}{3} \int_q \gamma_\mu S(p) \gamma_\nu T_{\mu\nu}(k) G(k^2), \quad (5)$$

where S is the full quark propagator, S_0 the corresponding bare quantity and Z_2 is the quark wave-function renormalisation. $T_{\mu\nu}(k)$ is the transverse projector and $G(k^2)$ is the effective gluon dressing. This function is modelled in the present approach in a way such that chiral symmetry breaking occurs while the axial-vector Ward-Takahashi identity (AXWTI), the $U(1)$ vector-WTI of QED and resummed one-loop perturbation theory are respected [10]. Consistent with the quark DSE in (5) is the meson

BSE

$$[\Gamma]_{rs}(P, k) = -Z_2^2 \frac{4}{3} \int_q [S(q_+) \Gamma(P, q) S(q_-)]_{ut} K_{tu,rs}(k - q), \quad (6)$$

where P is the meson momentum, k the relative quark momentum and $q_{\pm} = q \pm P/2$. The interaction kernel is defined as

$$K_{rs,tu}(k) = G(k^2) T_{\mu\nu}(k) [\gamma_{\mu}]_{rt} [\gamma_{\nu}]_{us}. \quad (7)$$

The latter two equations are intimately related by chiral symmetry, to give a dynamical breaking in accordance with Goldstone's theorem [11]. In addition we need the quark-photon vertex defined via the inhomogeneous BSE

$$[\Gamma_{\mu}]_{rs}(P, k) = Z_2 \gamma_{\mu} - Z_2^2 \frac{4}{3} \int_q [S(q_+) \Gamma_{\mu}(P, q) S(q_-)]_{ut} K_{tu,rs}(k - q), \quad (8)$$

which is the key to any calculation of electromagnetic properties of hadrons. The vertex features a vector-meson bound-state for time-like momenta P such that vector-meson dominance (VMD) is dynamically included. This gives e.g. important contributions to the pion charge radius which can be nicely described in the present approach [12].

3 Hadronic vacuum polarisation (HVP)

Here we present our results briefly, more details can be found in Ref. [13]. The central object for the HVP contribution is the hadronic tensor

$$\Pi_{\mu\nu}(p) = -Z_2 e^2 \int_q \text{Tr}[S(q_+) \Gamma_{\mu}(p, q) S(q_-) \gamma_{\nu}] , \quad (9)$$

which corresponds to the 1PI hadronic photon self-energy and involves the non-perturbative quark propagator (5) and the self-consistent quark-photon vertex (8). The tensor $\Pi_{\mu\nu}$ is transverse due to its WT

$$\Pi_{\mu\nu}(p) = \left(\delta_{\mu\nu} - \frac{p_{\mu} p_{\nu}}{p^2} \right) p^2 \Pi(p^2) , \quad (10)$$

which serves as a definition of the scalar function $\Pi(p^2)$. We use the renormalisation condition $\Pi_R(p^2) = \Pi(p^2) - \Pi(0)$ which corresponds to the usual physical QED on-shell scheme giving rise to $e(0) = e_{\text{physical}}$. Another quantity that is interesting in the present context is the Adler function

$$D(q) = -q^2 \frac{d\Pi_R(q^2)}{dq^2} . \quad (11)$$

The results for the Adler function from DSE's including five quark flavours is shown in fig. 2 and compared to a result from dispersion relations [9]. There is quite reasonable agreement at all momentum scales. Especially the deeply non-perturbative behaviour below 1 GeV is nicely reproduced. It is this regime, set by the muon mass, where the contribution to a_{μ} saturates.

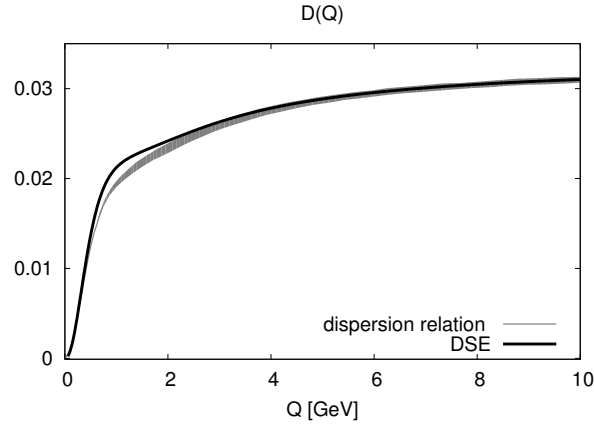


Figure 2: The Adler function obtained from DSE's using our parameter set II compared to a model independent result from dispersion relations.

[MeV]	$m_{u,d}$	m_s	m_π	m_K	$m_{\rho,\omega}$	m_ϕ
set I	3.7	85	138	495	740	1080
set II	11	72	240	477	770	1020

Table 1: Two choices for the light bare quark masses at $\mu^2 = (19 \text{ GeV})^2$ and the resulting meson masses (in MeV) in the pseudoscalar and vector meson sector. For the heavy quarks we always take $m_c = 827$ MeV and $m_b = 3680$ MeV which lead to good results for charmonia and bottomonia in the pseudoscalar and vector channel.

The contribution to the muon $g - 2$ can be obtained via [14]

$$a_\mu^{\text{HVP}} = \frac{\alpha}{\pi} \int_0^1 dx (1-x) \left[-e^2 \Pi_R \left(\frac{x^2}{1-x} m_\mu^2 \right) \right] , \quad (12)$$

where α is the fine-structure constant. In particular we use two different parameter sets. The standard parameter set where the u , d and s quark masses are fixed to the pseudo-scalar meson sector and another one where instead the vector-meson sector is used. This is summarized in table 1. The c and b quark mass functions are fixed to the charmonium and bottomonium vector meson states in all cases [15]. With these two parameter sets we obtain the following results

$$a_\mu^{\text{HVP,I}} = 7440 \times 10^{-11} , \quad a_\mu^{\text{HVP,II}} = 6760 \times 10^{-11} . \quad (13)$$

Comparing to the model independent result (3) we see that the standard set I deviates about 7 percent. We see the reason for this in the ρ mass which is about four percent too light, see tab. 1. The result with the physical ρ mass (set II) is indeed closer to the dispersion relation result. Taking the idea of changing the ρ mass a step further, we calculate the two flavour contribution $a_\mu^{\text{HVP}, N_f=2}$ as a function of the vector meson mass. Both are functions of the quark masses $m_{u/d}$. In addition we calculate $a_\mu^{\text{HVP}, N_f=2+1}$ where the strange contribution is just an additive constant since m_s remains fixed. The results are shown in fig. 3 where we compare to $N_f = 2$ results from the ETMC collaboration [16] (blue data) and $N_f = 2 + 1$ data (red) from the RBC-UKQCD collaboration [17]. The ordinate shows a_μ in units of $[10^{-11}]$. Our curves agree with both data within error bars which we take as a hint that the DSE/BSE approach in the present truncation captures the relevant degrees of freedom. For HVP this seems to be more than anything else the vector meson as would be expected from VMD estimates [18]. For a detailed discussion see Ref. [13].

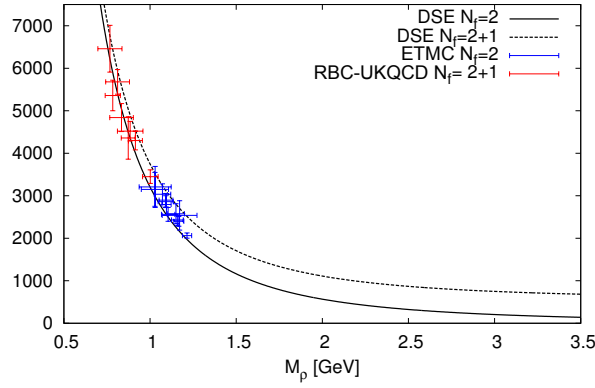


Figure 3: The $N_f = 2$ and $N_f = 2 + 1$ flavour contribution to a_μ^{HVP} in units of $[10^{-11}]$ as a function of the vector meson mass. For the latter case the s contribution is kept constant. The two DSE curves are compared to lattice results for the two- and three flavour case respectively. The blue data is $N_f = 2$ data from ETMC [16] and the red data is $2 + 1$ from RBC-UKQCD [17].

4 Hadronic light-by-light scattering (LBL)

In the present section we discuss the LBL contribution. Within the framework of DSE's the hadronic four-point function, that is the essential ingredient here, has a description that is consistent with the one for HVP shown earlier (9). We presented this representation in Refs. [19, 20] where also more details can be found. There we also elaborate on the resonance expansion of the quark anti-quark T-matrix that is used for the results presented here. To this end we arrive at an approximate description of the full four-point function

$$\Pi_{\mu\nu\alpha\beta} = \text{QL} + \text{meson exchange}, \quad (14)$$

that consists of the non-perturbatively dressed quark loop diagram (QL) as well as a pseudo-scalar meson-exchange contribution that takes into account the π^0 , η and η' mesons. This picture is very similar to the one obtained in hadronic models and EFT [14, 21, 22, 23, 24, 25, 26].

In order to obtain the contribution to a_μ^{LBL} from the four-point function we define

$$ie\tilde{\Gamma}_{\rho\mu} = \int_{q_1} \int_{q_2} D_{\epsilon\nu}(q_1) D_{\delta\alpha}(q_2) D_{\gamma\beta}(q_3) (ie\gamma_\gamma) S(p_1) (ie\gamma_\delta) S(p_2) (ie\gamma_\epsilon) \left[(ie)^4 \tilde{\Pi}_{(\rho)\mu\nu\alpha\beta}(q_1, q_2, q_3) \right], \quad (15)$$

from which the anomaly can be obtained via [27]

$$a_\mu = \frac{1}{48m_\mu} \text{Tr} \left[(i\not{P} + m_\mu) [\gamma_\sigma, \gamma_\rho] (i\not{P} + m_\mu) \tilde{\Gamma}_{\sigma\rho} \right] \Big|_{k \equiv 0}, \quad (16)$$

where P is the muon momentum. Here S and $D_{\mu\nu}$ are perturbative muon and photon propagators and the definition $\tilde{\Pi}_{(\rho)\mu\nu\alpha\beta} = \partial_\rho \Pi_{\mu\nu\alpha\beta}$ has been used with the hadronic four-point function $\Pi_{\mu\nu\alpha\beta}$.

For the pseudo-scalar (PS) meson pole contribution we need the PS $\gamma\gamma$ form factor

$$\Lambda_{\mu\nu}^{\text{PS}\gamma\gamma}(k_1, k_2) = 2e^2 N_c \int_k \text{Tr} \left[i\hat{Q}_e \Gamma_\nu(k_2, p_{12}) S_F(p_2) \hat{\Gamma}^{\text{PS}}(p_{23}, P) S_F(p_3) i\hat{Q}_e \Gamma_\mu(k_1, p_{31}) S_F(p_3) \right], \quad (17)$$

that is defined as a non-perturbative quark triangle that involves the quark (5), the quark-photon vertex (8) and the meson amplitude (6) called $\hat{\Gamma}^{\text{PS}}$ here. The meson momentum is P , $k_{1/2}$ are the photon momenta, p_i the momenta of the quarks and $p_{ij} = (p_i + p_j)/2$. From the form factor together with a bare meson propagator we obtain the resonant part that is shown in Eq. (14). Details concerning the flavour content of the meson as well as the necessary meson off-shell prescription can be found in [20]. Once the form factor is known the contribution to $g - 2$ can be obtained along the lines explained in [24]. Our result for the PS meson exchange contribution (π^0, η, η') is

$$a_\mu^{\text{LBL;PS}} = (80.7 \pm 12.0) \times 10^{-11}, \quad (18)$$

where the error is dominantly an estimate of the systematic model uncertainty.

For the QL contribution we take the full quark propagator (5) together with the Ball-Chiu (BC) vertex construction [28]

$$\Gamma_\mu^{\text{BC}}(P, k) = \gamma_\mu \Sigma_A + 2\not{k} k_\mu \Delta_A + i k_\mu \Delta_B, \quad (19)$$

where the symbols

$$\Sigma_F = \frac{F(k_+^2) + F(k_-^2)}{2}, \quad \Delta_F = \frac{F(k_+^2) - F(k_-^2)}{k_+^2 - k_-^2}, \quad (20)$$

have been used. This substructure of the fully self-consistent vertex (8) is dictated by the vector WTI. Defining the four-point function from the quark-loop as in Eq. (14), taking the derivative and using Eqs. (15) and (16) we obtain

$$\begin{aligned} a_\mu^{\text{LBL;quarkloop (bare vertex)}} &= (61 \pm 2) \times 10^{-11} \\ a_\mu^{\text{LBL;quarkloop (1BC)}} &= (107 \pm 2) \times 10^{-11} \\ a_\mu^{\text{LBL;quarkloop (BC)}} &= (176 \pm 4) \times 10^{-11}, \end{aligned} \quad (21)$$

where the first result uses a bare quark-photon vertex and 1BC only has the γ_μ part of the dressed vertex, see (19). In this calculation we included the flavours u, d, s and c . It can clearly be seen that the vertex dressing causes quite some enhancement especially when all three structures of the BC vertex are used (third case). The error given here is numerical. Clearly, this result is preliminary, since the transverse structure of the vertex including important contributions from vector mesons is missing. We have estimated these contributions in [20], however a full calculation is absolutely mandatory and well under way.

5 Discussion

We presented results for the HVP as well as for the LBL contribution to the muon $g - 2$ obtained within the framework of DSE's. We saw that in the case of HVP our results for a_μ^{HVP} (13) reproduce model independent dispersion relations on the less-than-ten-percent level. We see no principal reason why this should not be the case also for a full LBL calculation. Indeed, our result for the pseudo-scalar meson-exchange contribution to LBL (18) is in the ballpark of the results obtained within other approaches [14, 21, 22, 23, 24, 25, 26] (see [4] for an overview). For the quark loop contribution to LBL we take our BC result from (21) and guesstimate the missing contributions from the above mentioned transverse components of the quark-photon vertex, see [20] for details. Thus we arrive at the value $a_\mu^{\text{LBL;quarkloop (BC+transverse)}} = (136 \pm 79) \times 10^{-11}$, where the large error takes into account the uncertainties due to the estimate. Putting all contributions together we obtain

$$a_\mu^{\text{LBL}} = (217 \pm 91) \times 10^{-11}, \quad (22)$$

As mentioned already our result for LBL hints towards a larger contribution and thus to a smaller deviation between theory and experiment as compared to Eqs. (1, 2). Besides implementing the full quark-photon vertex inside the quark-loop, another important next step is to overcome the resonance approximation of LBL. In general, we believe to have shown that a full calculation from the DSE approach can be expected to mark a clear step forward for the case of the hadronic light-by-light scattering contribution to the muon $g - 2$.

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